Lezioni dottorali, Università degli Studi di Napoli "Parthenope"

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Nonlinearity, Tipping Points & Chaos in the Climate Sciences

Michael Ghil

Ecole Normale Supérieure, Paris, and
University of California, Los Angeles

Please visit these sites for more info.

https://dept.atmos.ucla.edu/tcd

http://www.environnement.ens.fr/

Overall Outline

➔ **Lecture I: EBMs(+), paleoclimate & "tipping points"**

- **Lecture II: The wind-driven ocean circulation**
- **Lecture III: Advanced spectral methods–SSA(±) *et al*.**
- **Lecture IV: Nonlinear & stochastic models—RDS(❖)**

(+) EBM = Energy balance model
(±) SSA = Singular-spectrum analysis
(❖) RDS = Random dynamical system

Dottorato di ricerca
“Fenomeni e rischi ambientali”
Serie di quattro lezioni dal Professore M. Ghil

NONLINEARITY, TIPPING POINTS AND CHAOS IN THE CLIMATE SCIENCES

con il seguente orario:

Venerdi 21,	ore 11-13,	Aula 6	(I piano Sud)
Lunedi 24,	ore 11-13,	Aula 17	(III piano No
Mercoledi 26,	ore 15-17,	Aula 1	(I piano)
Giovedi 27,	ore 15-17,	Aula 1	(I piano)

Motivation

- The ***climate system*** is highly ***nonlinear and*** quite ***complex***.
- The system's ***major components*** — the atmosphere, oceans, ice sheets — ***evolve*** on many time and space scales.
- Its ***predictive understanding*** has to rely on the system's physical, chemical and biological modeling,

  but also on the thorough mathematical analysis of the models

  thus obtained: ***the forest vs. the trees***.
- The ***hierarchical modeling*** approach allows one to

  give proper weight to the understanding provided by the

  models *vs*. their realism: back-and-forth between

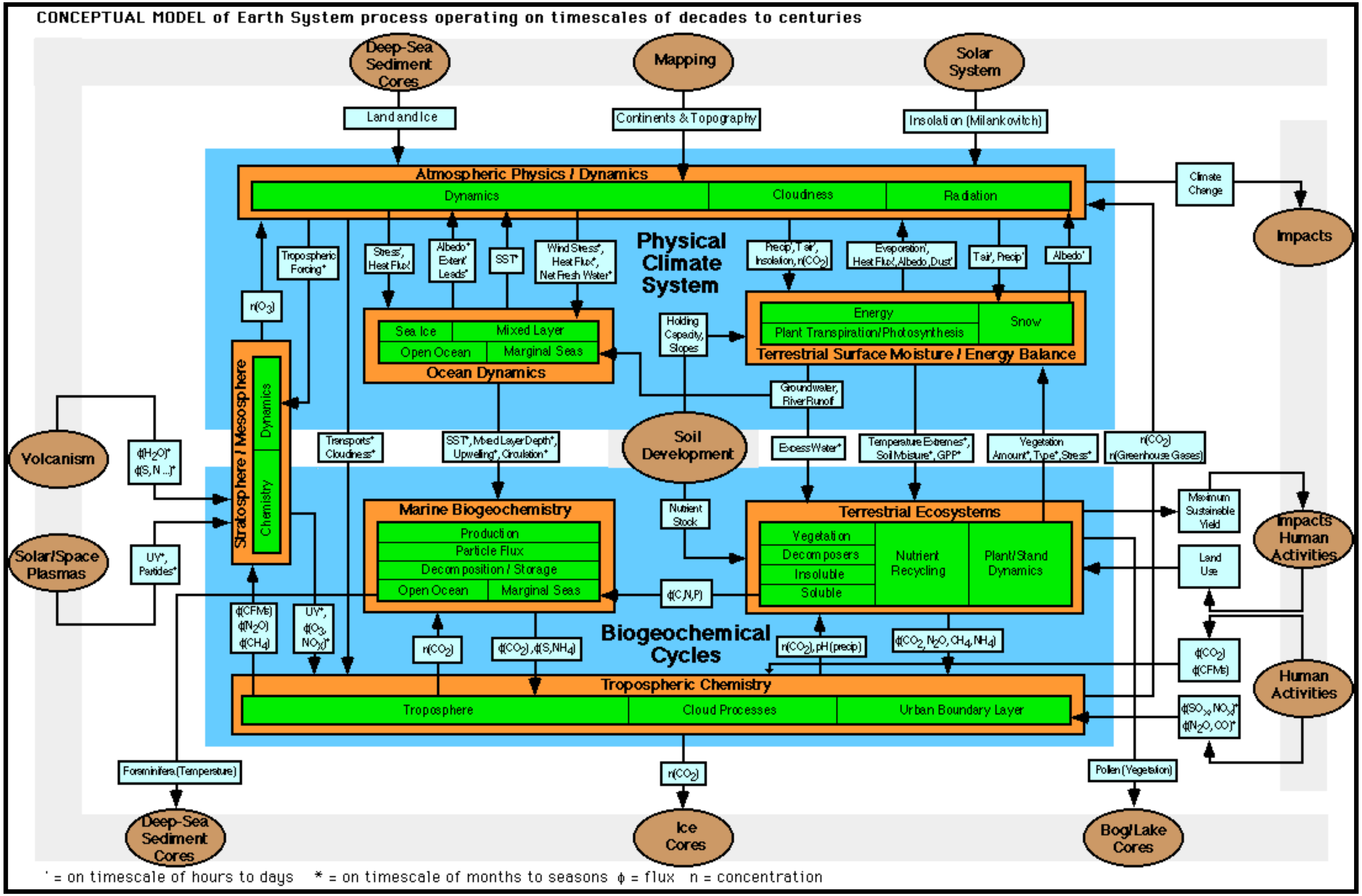
  "toy" (conceptual) and ***detailed*** ("realistic") ***models***,

  and between ***models*** and ***data***.
- How do we disentangle ***natural variability*** from ***the anthropogenic forcing***: ***can we & should we, or not?***

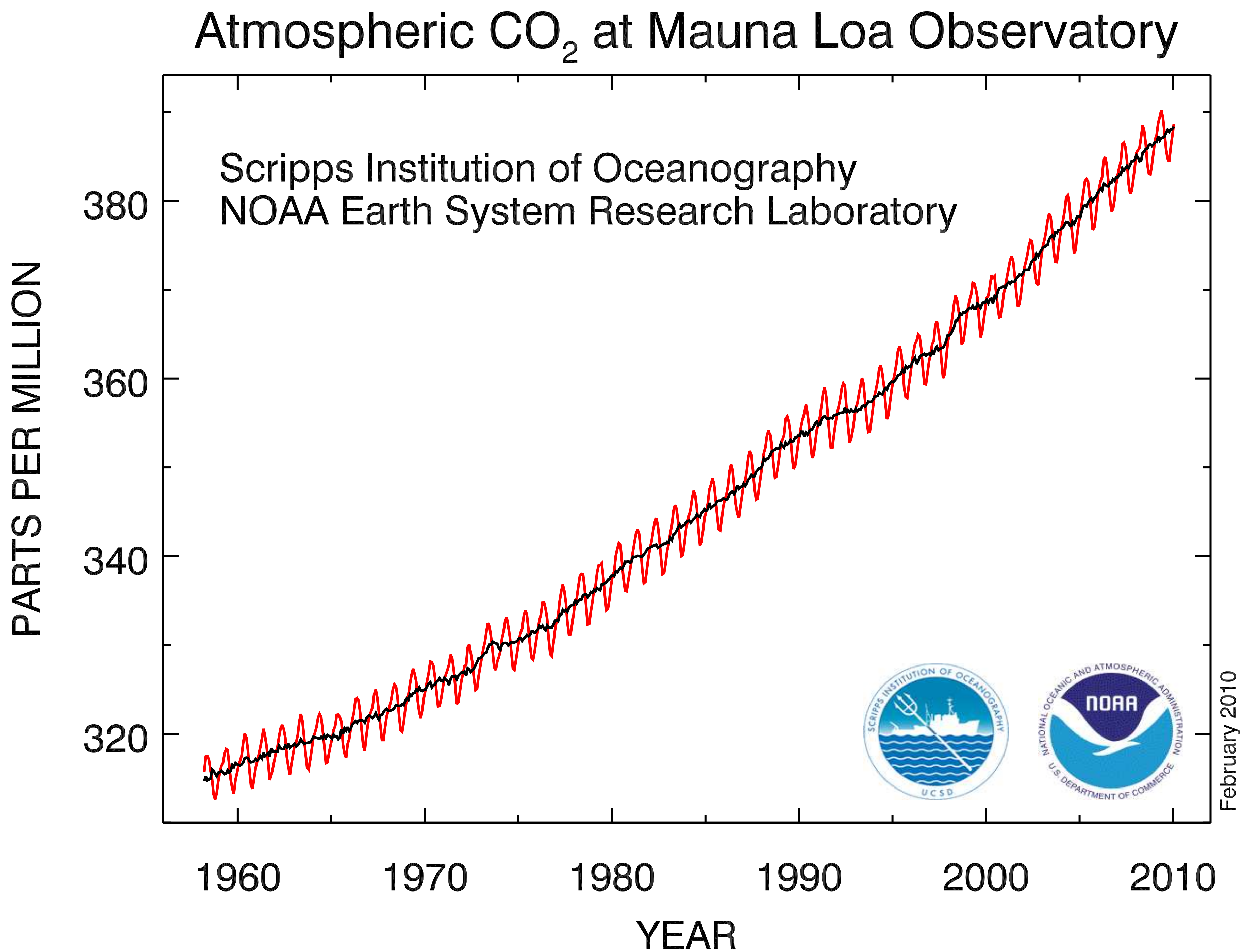
Outline: Radiative equilibrium & EBMs

1. Climate system & climate sensitivity
   - climate subsystems & other complexities
   - climate sensitivity & IPCC results
2. Radiative equilibrium
   - global (0-D) radiative equilibrium
   - latitude-dependent (1-D) radiative equilibrium
3. EBMs in 0-D (global)
   - equilibria & their linear stability
   - multiple stationary climates
4. Elementary bifurcation theory in 0-D
   - saddle-node bifurcations: multiple equilibria & their stability
   - nonlinear stability and variational principle
   - bistability and hysteresis
5. EBMs in 1-D (latitude-dependent)
   - applying all of the above
   - snowball Earth

F. Bretherton's "horrendogram" of Earth System Science



Earth System Science Overview, *NASA Advisory Council*, 1986

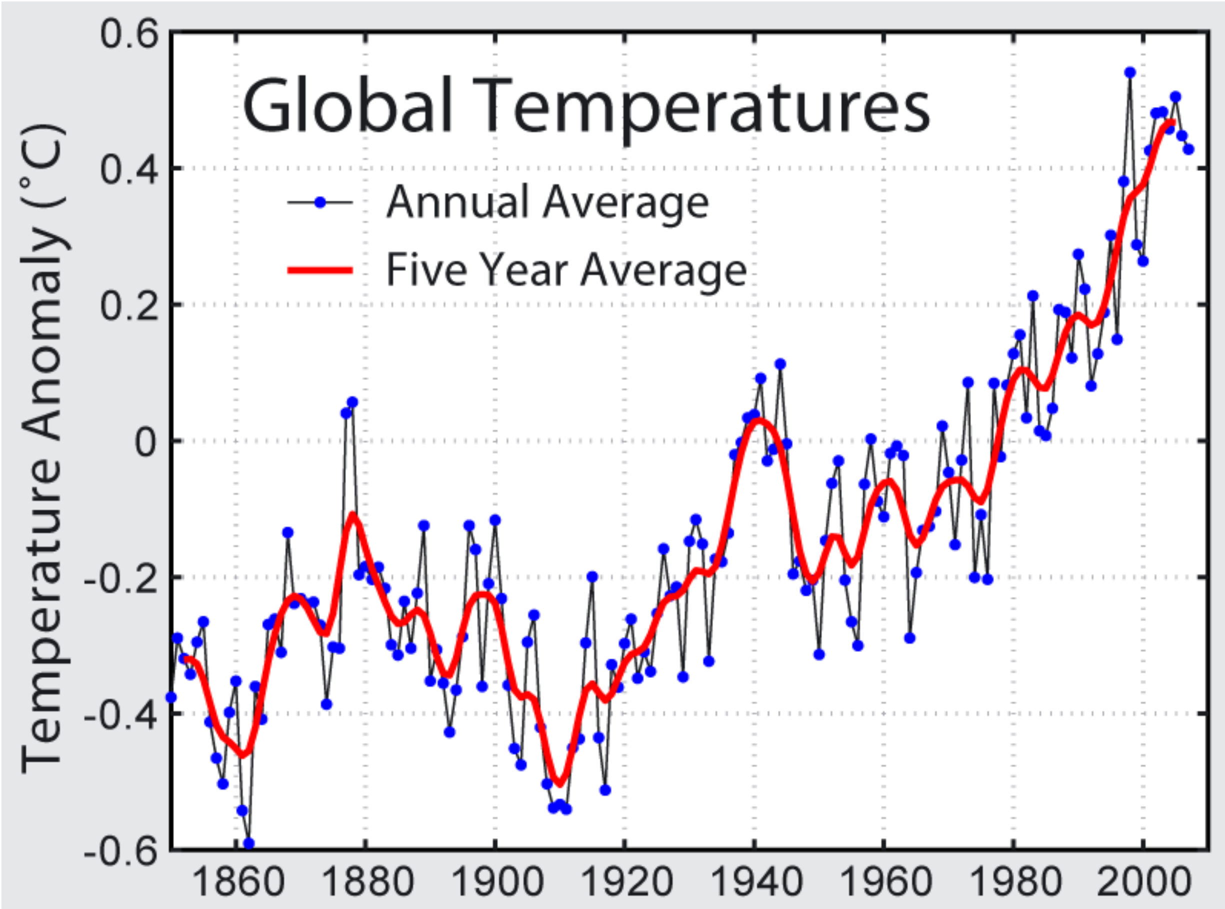


Atmospheric CO_2 at Mauna Loa Observatory
Scripps Institution of Oceanography
NOAA Earth System Research Laboratory
PARTS PER MILLION
320
340
360
380
1960
1970
1980
1990
2000
2010
YEAR
February 2010

Temperatures and GHGs

Greenhouse gases (GHGs) go up,
temperatures go up:

It's gotta do with us, at least a bit,
doesn't it?



Wikicommons, from
Hansen *et al*. (*PNAS*, 2006);
see also http://data.giss.nasa.gov/
gistemp/graphs/

Global warming and its socio-economic impacts

Temperatures rise:

- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, *i.e.*, it depends on the accuracy and reliability of the forecast …

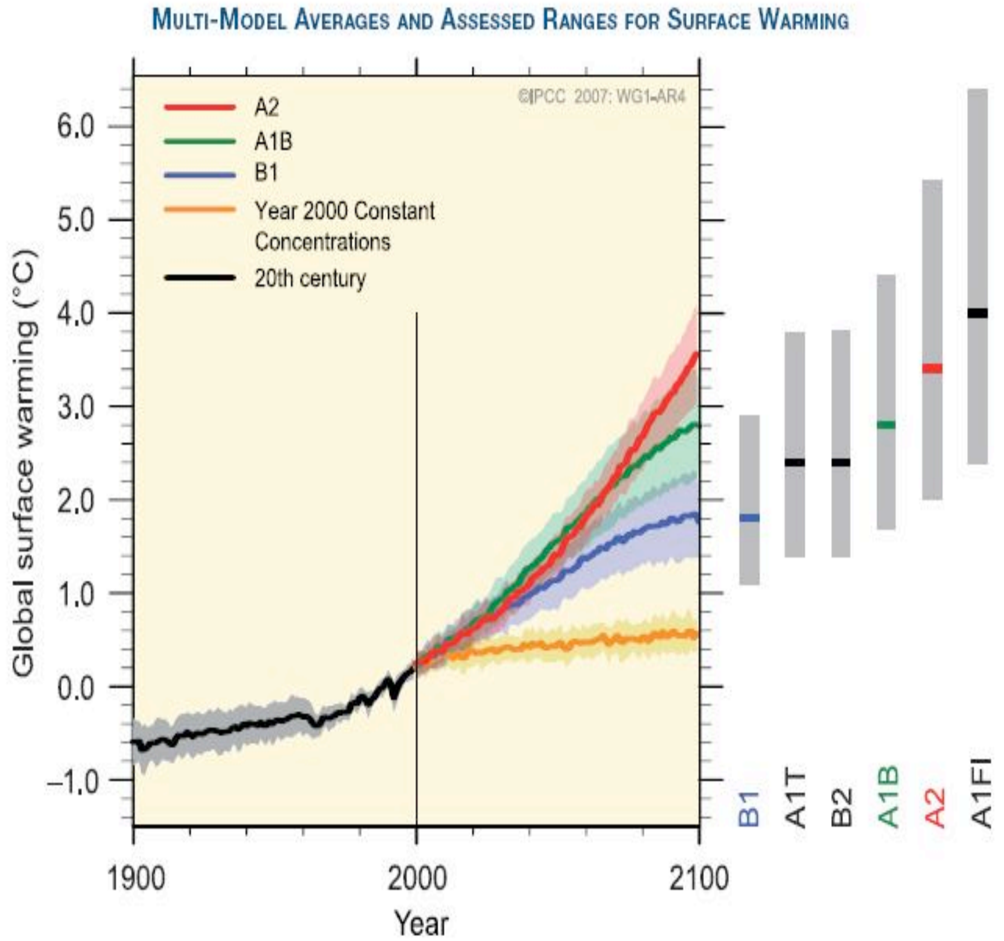


Figure SPM.5. *Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ±1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the* **likely** *range assessed for the six SRES marker scenarios. The assessment of the best estimate and* **likely** *ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. {Figures 10.4 and 10.29}*

Source : IPCC (2007), *AR4, WGI, SPM*

Global warming and its socio-economic impacts– II

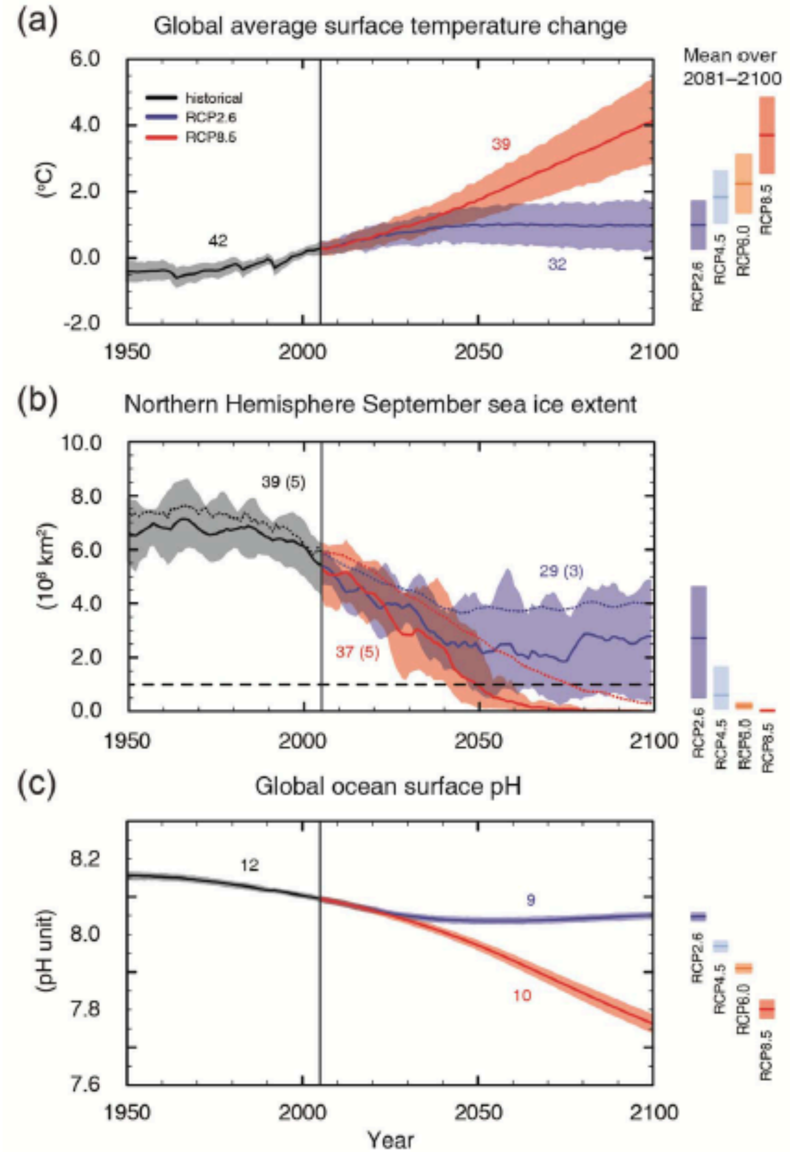
Temperatures rise:

- What about impacts?
- How to adapt?

AR5 vs. AR4

A certain air of *déjà vu*:
GHG "scenarios" have been replaced by "representative concentration pathways" (RCPs), more dire predictions, but the uncertainties remain.

Source : IPCC (2013), AR5, WGI, SPM



Climate and Its Sensitivity

Let's say CO_2 doubles:

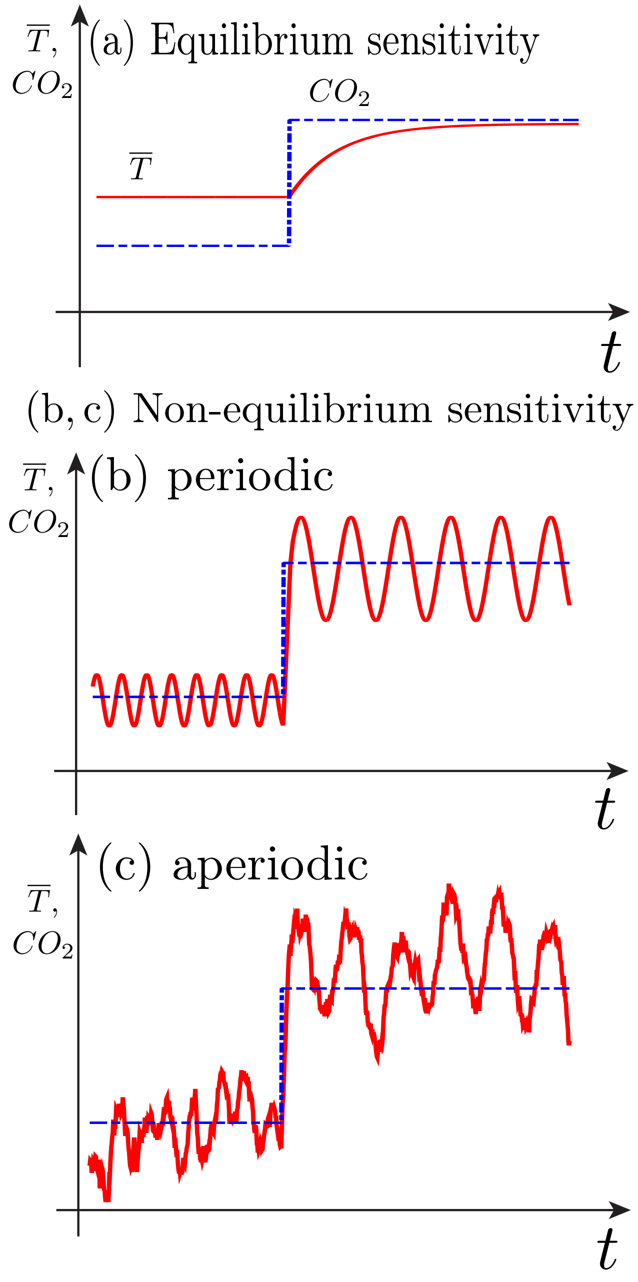
How will "climate" change?

1. Climate is in stable equilibrium (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

2. Climate is purely periodic; if so, mean temperature will (maybe) shift gradually to its new equilibrium value. But how will the period, amplitude and phase of the limit cycle change?

3. And how about some "real stuff" now: chaotic + random?

Ghil (in *Encycl. Global Environmental Change*, 2002)



Outline: Radiative equilibrium & EBMs

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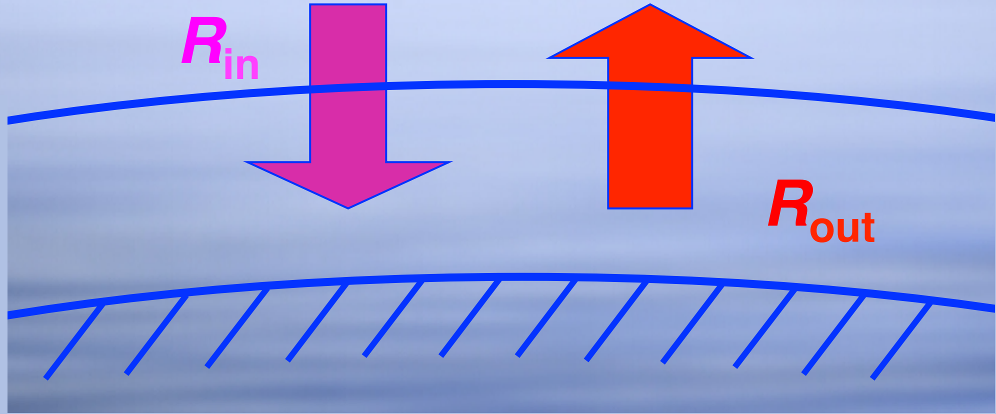
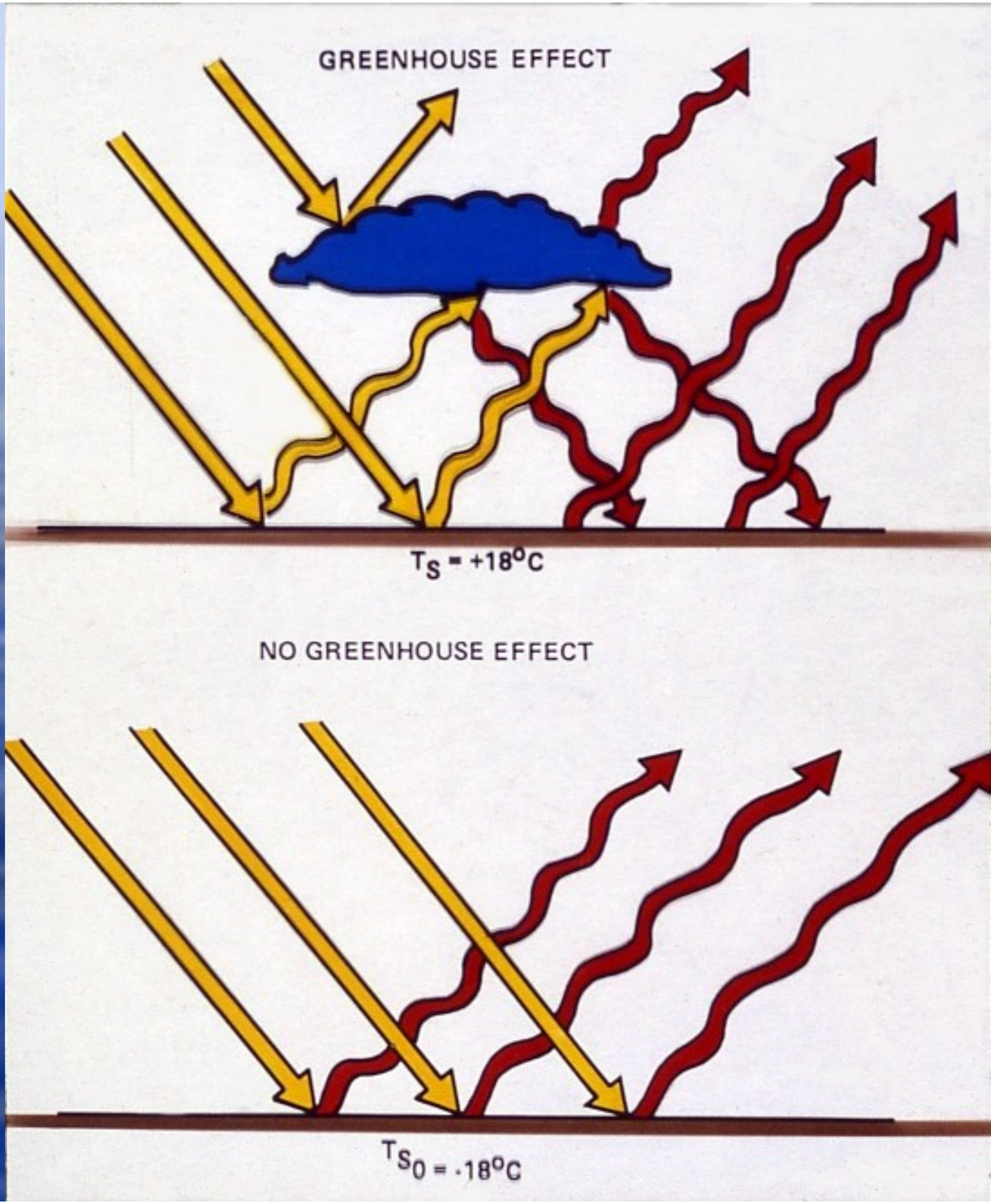
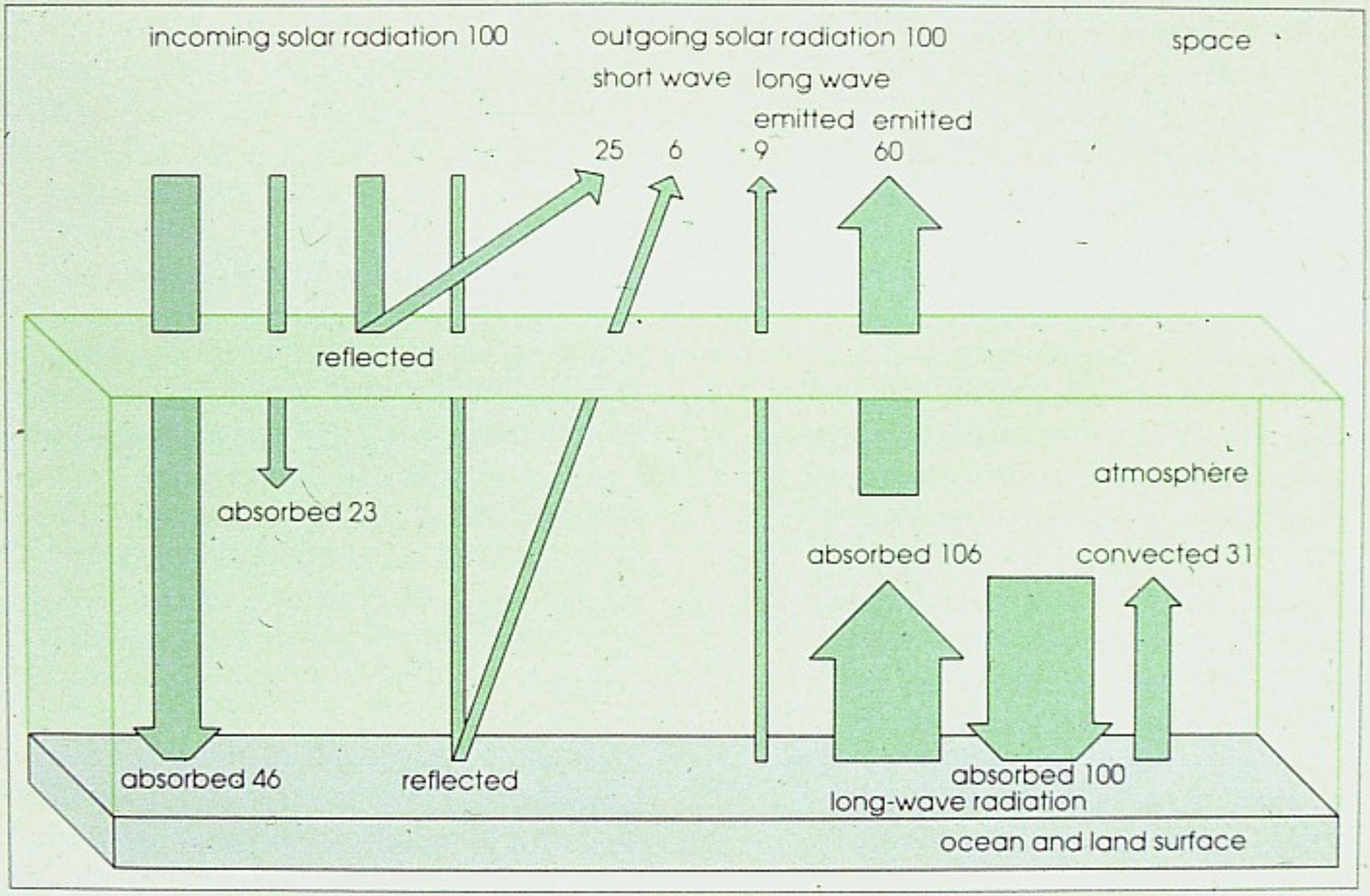
Radiative balance



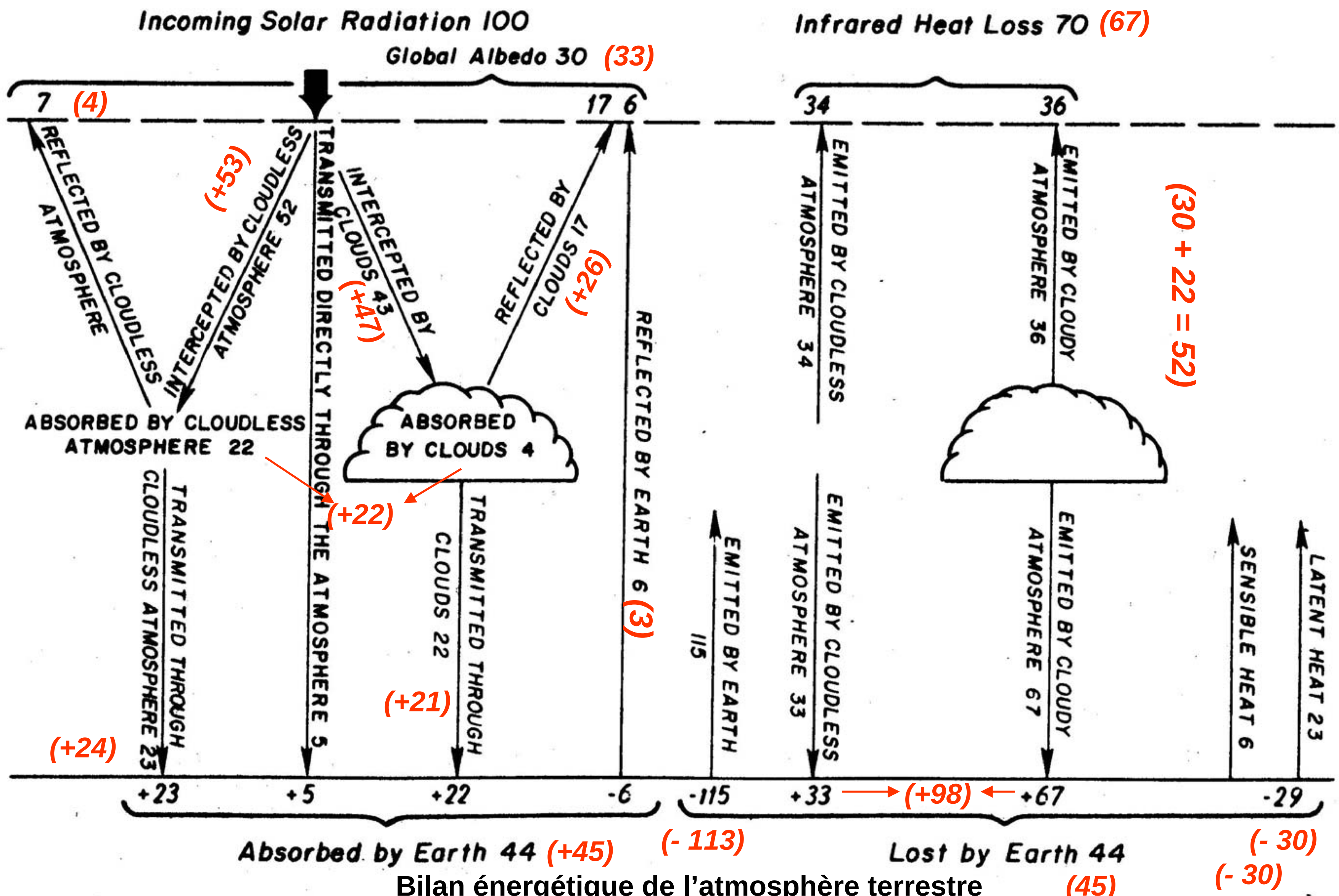
Long-term equilibrium between incident (solar, ultra-violet + visible) radiation R_{in} and outgoing (terrestrial, infrared) radiation R_{out} dominates climate.

Refs. **[1] Egyptian scribe** (3000 B.C.) :
''The Sun heats the Earth,'' *Rosetta stone*, ll. 13–17.
[2] **Herodotus** (484 - cca. 425 B.C.)


GREENHOUSE EFFECT
$T_S = +18^oC$
NO GREENHOUSE EFFECT
$T_{S_0} = -18^oC$



incoming solar radiation 100
outgoing solar radiation 100
space
short wave
long wave
emitted
emitted
25
6
9
60
reflected
absorbed 23
atmosphere
absorbed 106
convected 31
absorbed 46
reflected
absorbed 100
long-wave radiation
ocean and land surface



Bilan énergétique de l'atmosphère terrestre

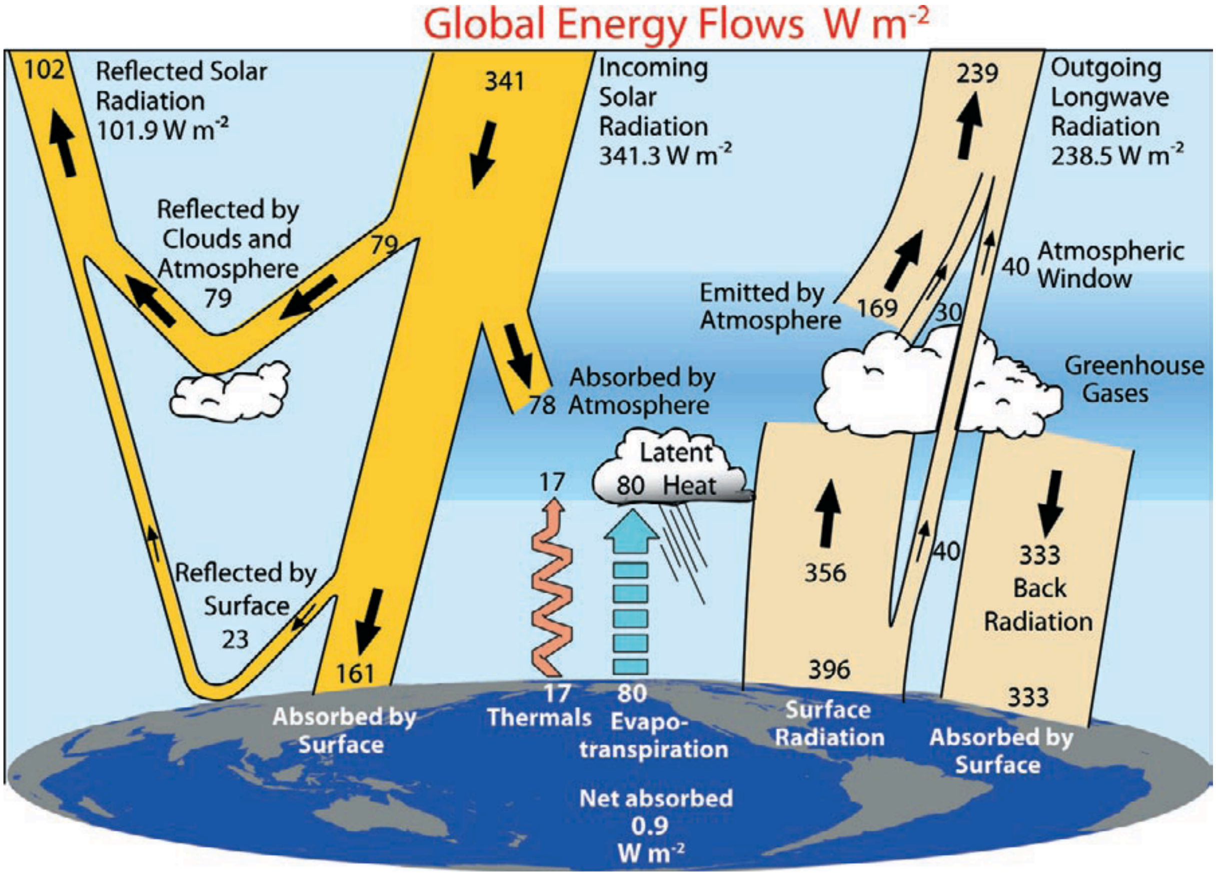
Valeurs en rouge: cf. figure précédente.

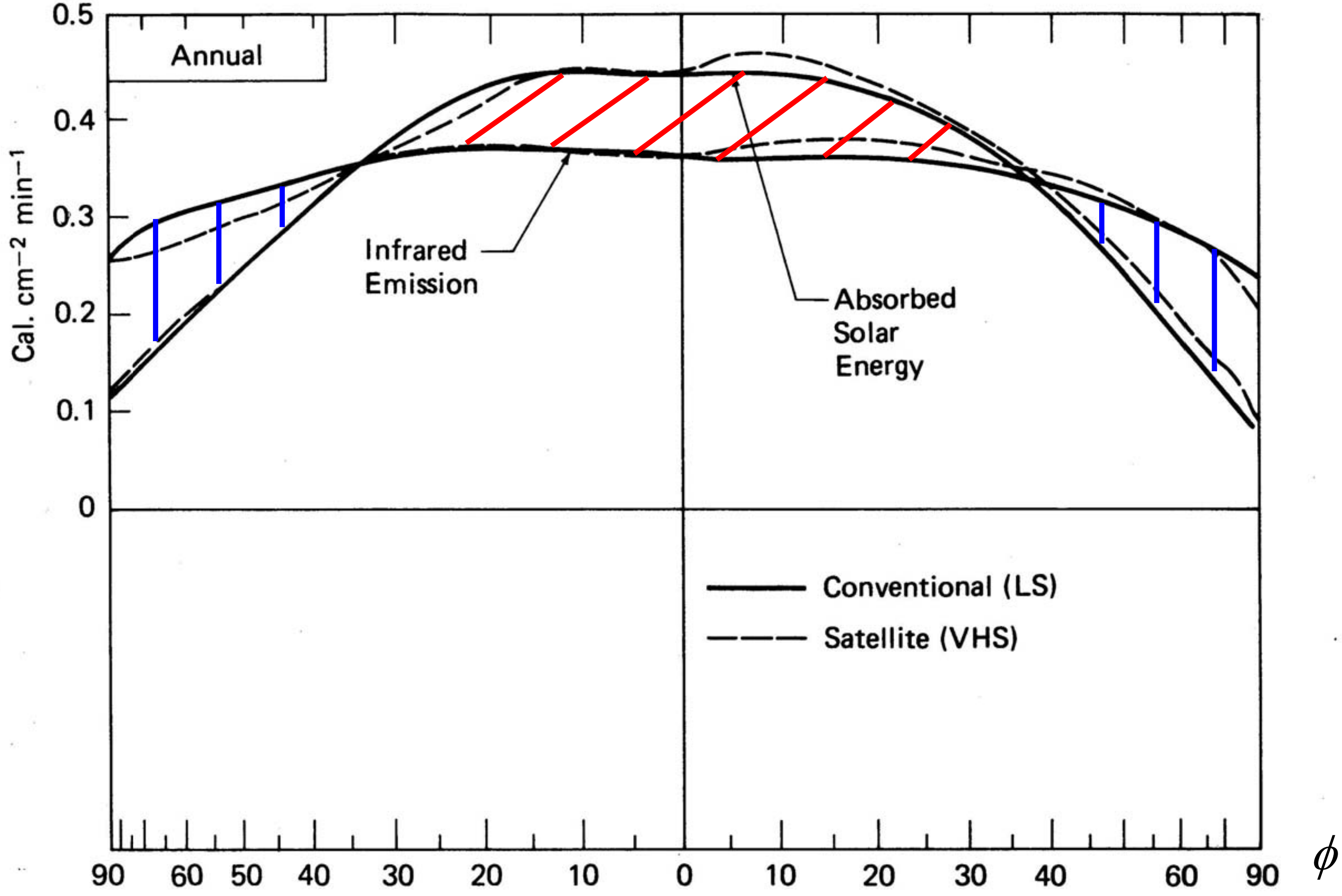
D'après **Kuo-Nan Liou, 1980:** ***An Introduction to Atmospheric Radiation*** **(fig. 8.19)**

Earth's Global Energy Budget

K.E. Trenberth, J.T. Fasullo & J. Kiehl, 2009,

Bull. Amer. Meteorol. Soc., **90**(3), 311–323.



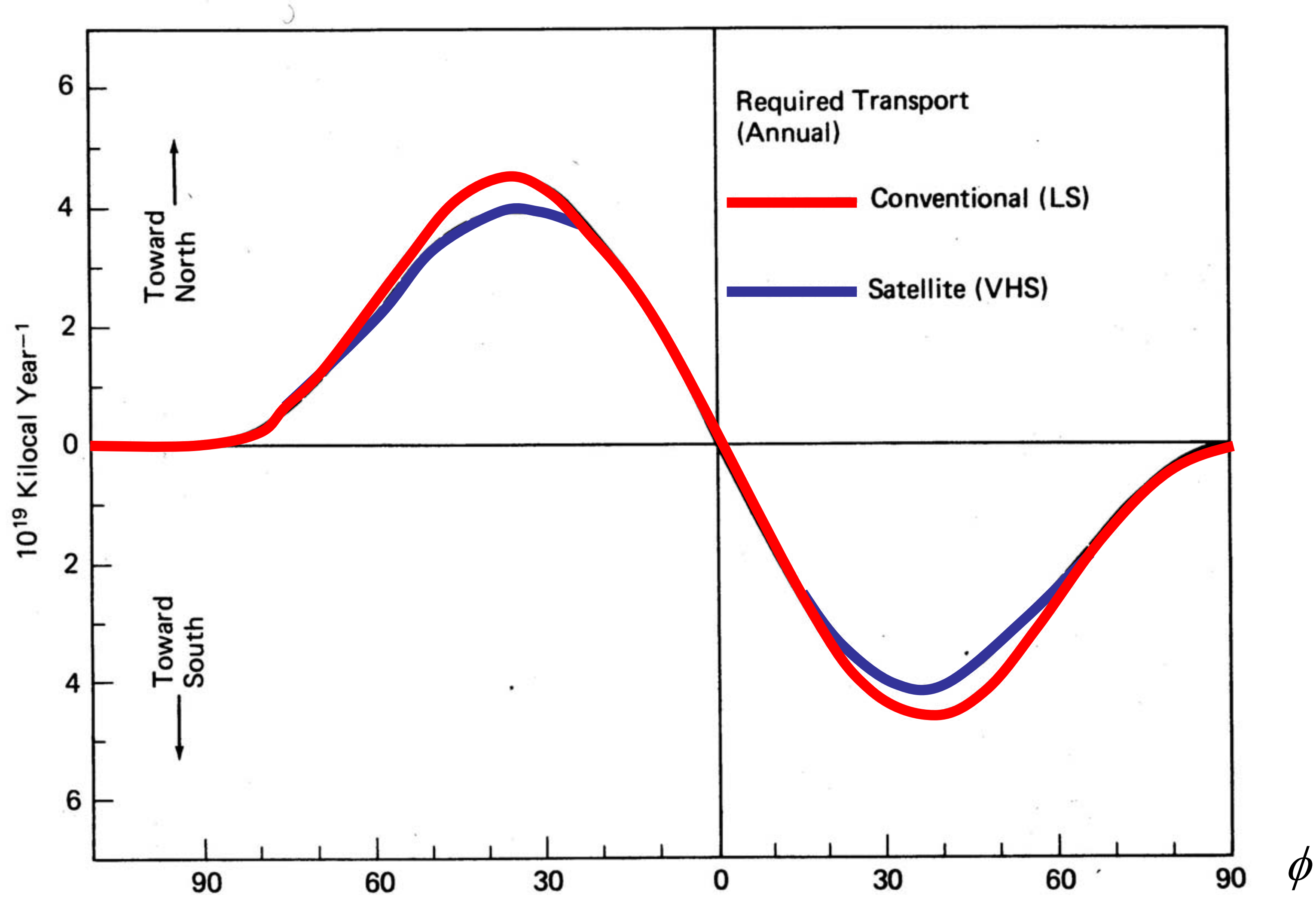


Bilan radiatif du système Terre-atmosphère en fonction de la latitude

$$R_i(\phi) - R_o(\phi) = \frac{1}{a.\cos(\phi)} \frac{\partial F}{\partial \phi}$$

Avec a — le rayon de la Terre et

F — le flux atmosphérique et océanique de chaleur



Transport atmosphérique et océanique d'énergie en fonction de la latitude

$$F(\phi) = a\int_0^{\phi} \cos(\varphi)\left[R_i(\varphi) - R_o(\varphi)\right]d\varphi$$

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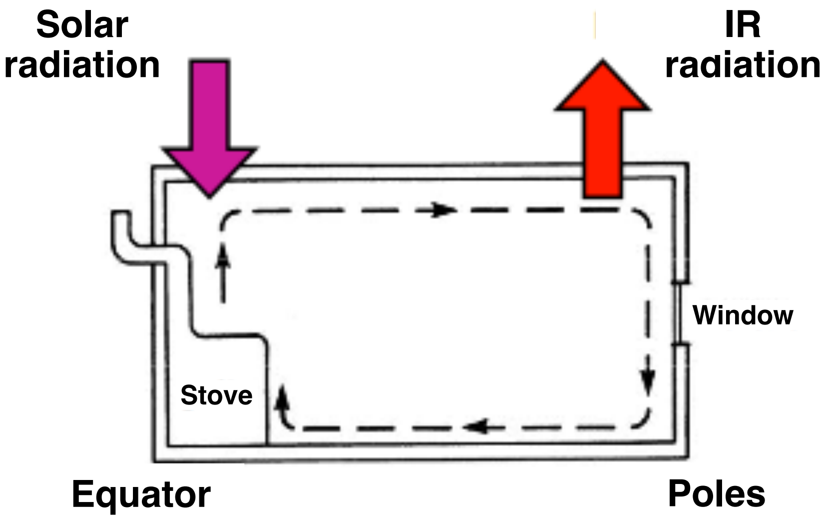
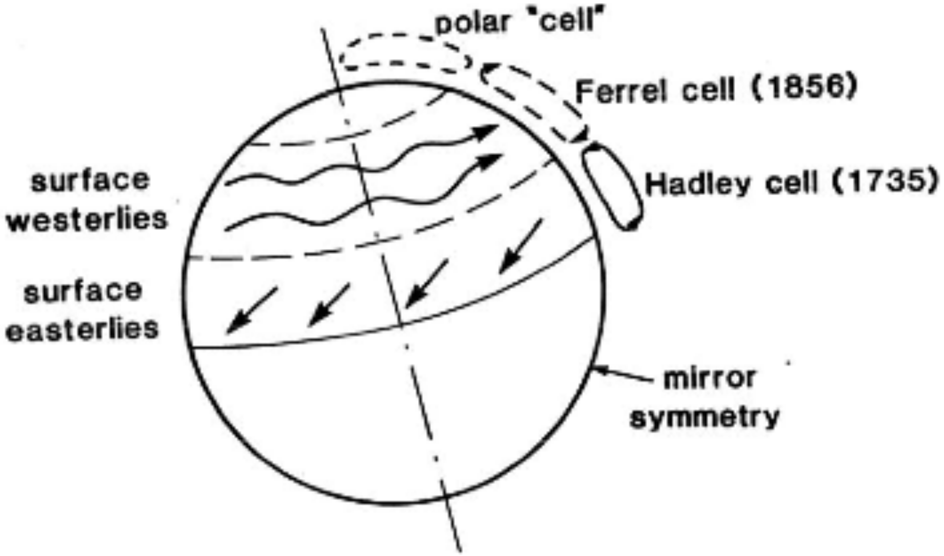
The mean atmospheric circulation

Direct Hadley circulation



Idealized view of the atmosphere's global circulation.*

Observed circulation



Schematic diagram of the atmospheric global circulation.*

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*From **Ghil and Childress (1987), Ch. 4**

Energy-balance models (EBMs)

$$C\frac{\partial T}{\partial t} = R_i - R_o + D$$

C — local calorific capacity

T — local surface temperature

R_i — incident solar radiation

R_o — terrestrial radiation towards space

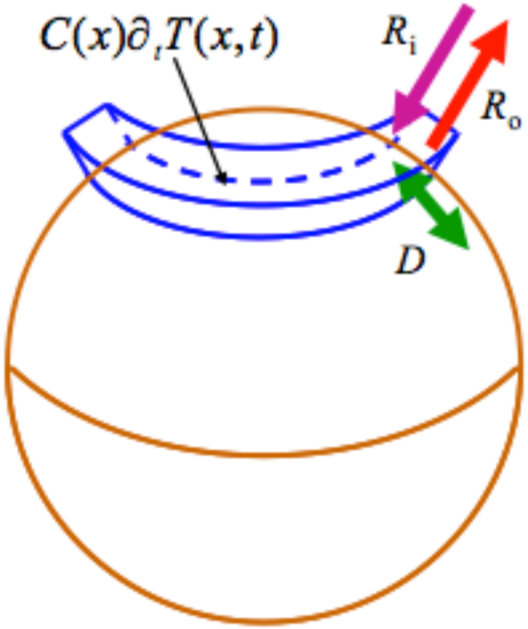
D — heat redistribution ('diffusion')

Comments:

1. a space variable (x) is maximum

2. C, R_i, R_o and D have to be calculated ("parameterized") according to $T = T(x,t)$

3. model's main characteristic:
$$R_i = Q(x)\,\{1 - \alpha(x,T)\}$$
with α the local albedo



0-D version (averaged over the globe)

$$C\frac{\partial \bar{T}}{\partial t} = R_i - R_o = Q\left\{1 - \alpha(\bar{T})\right\} - \sigma \bar{T}^4 m(\bar{T})$$

$\bar{T}$ — average surface temperature

t — time (in thousands of years)

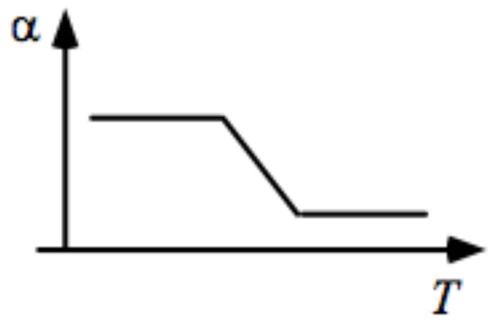
Q — incident solar flux

α — albedo

C — calorific capacity

σ — Stefan–Boltzmann constant

m — greenhouse effect factor



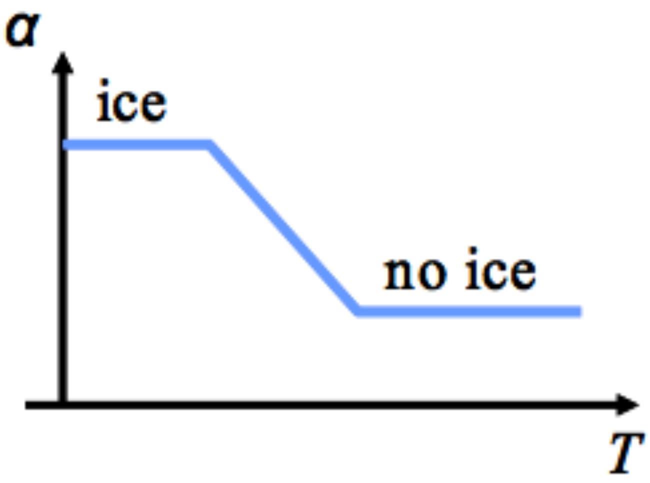
Comments:

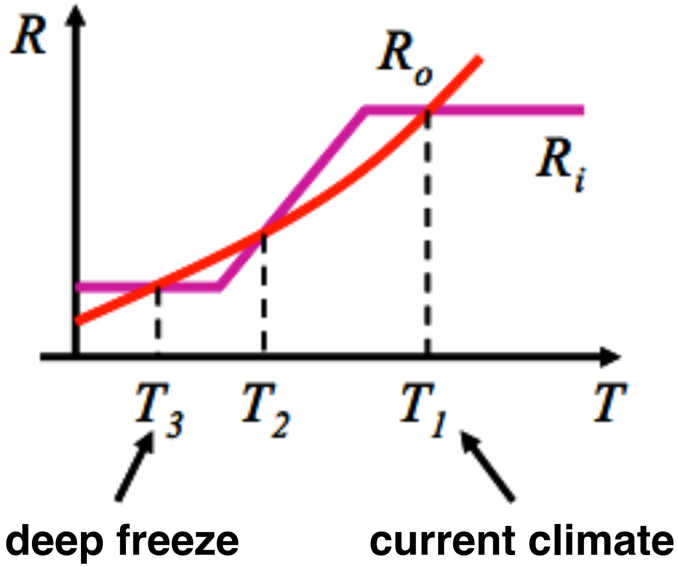
α depends on the ice and snow cover, on cloud cover, etc. (implicit variables). All is parameterized as a function of the explicit variable $\bar{T}$.

0-D EBM, I: Model solutions

We want to write T as: $T = T(t; T_0, Q, c, ...)$

Stationary solutions: $Q\{1 - \alpha(T)\} - \sigma T^4 = 0$





What happens if the sun "blinks" and $T = T_1 + \Delta T$?

We have to go back to the original equation, which depends on time.

0-D EBM, II: Stability condition

$$C\partial_t T = R_i - R_o = f(T)$$

$$R_i = Q\{1 - \alpha(T)\}$$
$$R_o = A + BT$$

We set $T = T_j + \theta$:

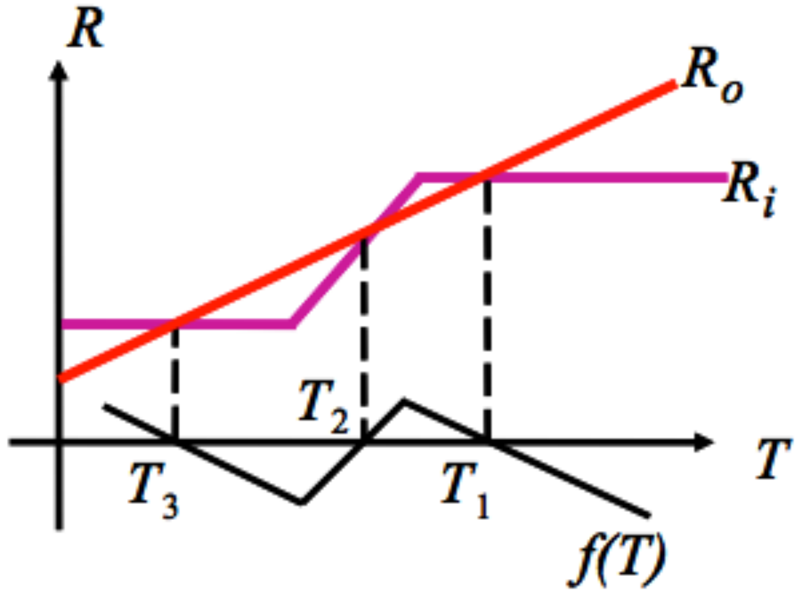
$f(T_j) = 0$,
$$f(T) = f(T_j) + f'(T_j)\theta + ...$$

Let's define $\lambda_j \equiv f'(T_j)/c$

$$\partial_t\theta = \lambda_j\theta \Rightarrow \theta = e^{\lambda_j t}\theta_0$$

If $\lambda_j < 0$ stable;

if $\lambda_j > 0$ unstable.

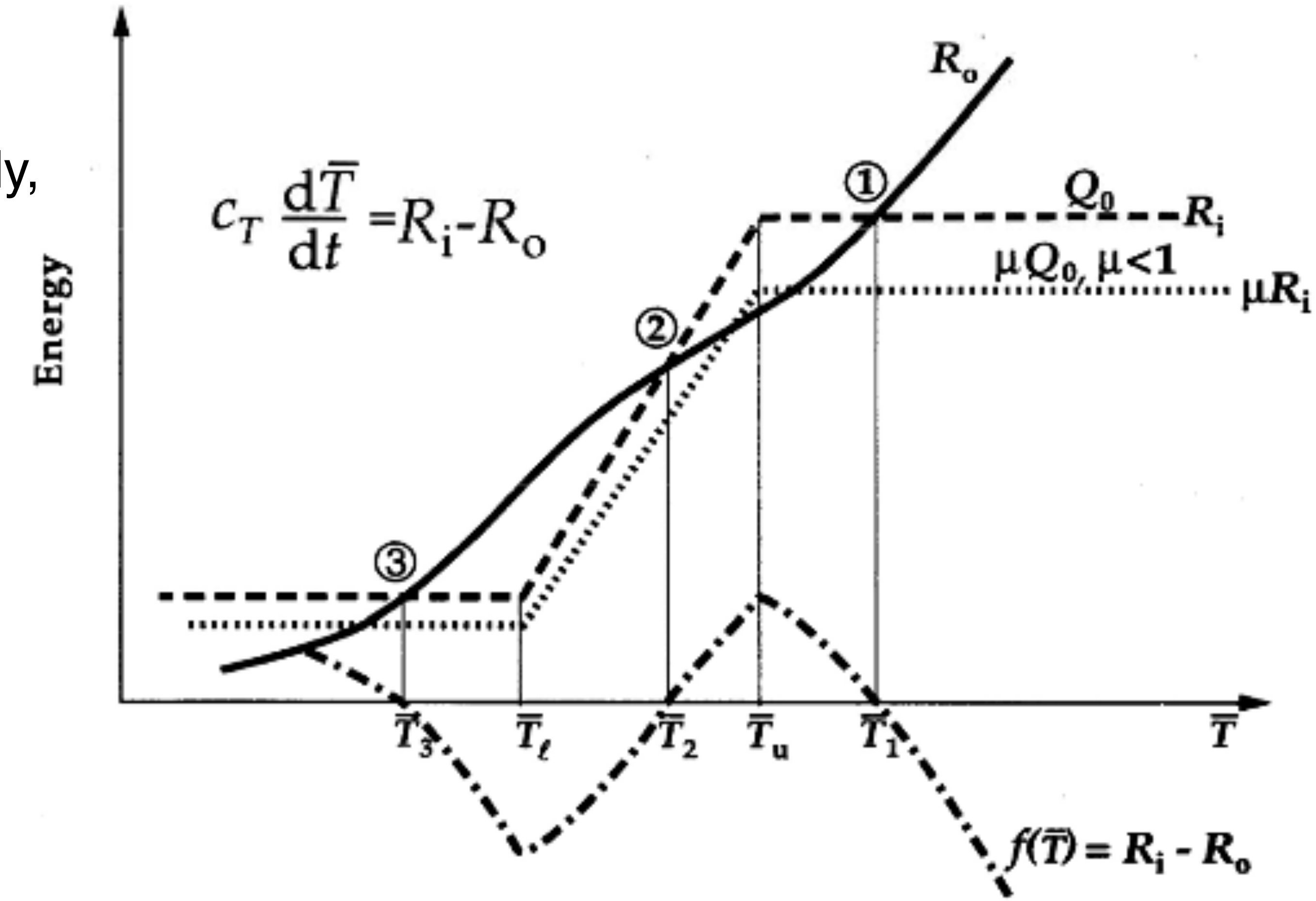


Comment: in the 1-D case $\lambda_j \to \lambda_j^{(0)}$;
$\lambda_j \sim 1/c$

0-D EBM, III: Changes in parameters

What happens if the insolation parameter μ changes, i.e., the "solar constant" changes? This may represent a change in solar luminosity, orbital parameters or in the optical properties of the atmosphere.

❖The model's three "climates" shift
in value and, possibly,
in number.



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Outline, Tipping Points I

Elementary Bifurcation Theory and Variational Principle

1. Fixed Points
   - linear stability
   - non-linear stability and attractor basins
2. Saddle-node bifurcations
   - multiple branches of stationary solutions
   - linear stability
3. Bifurcations in 1-D
4. Non-linear stability and variational principle
   - variational principle in 0-D
   - variational principle in 1-D
5. Bistability and hysteresis

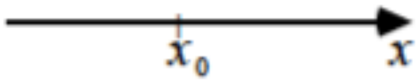
1. Fixed points, I

We start with a scalar ordinary differential equation (ODE)

$$\dot{x} = f(x;\mu)$$

depending on the parameter μ.

Linear stability, $\mu = 1$.



$$f(x_0) = 0 \Rightarrow \dot{x} = 0 \Rightarrow x \equiv x_0$$ – *Fixed point (FP)*

Consider an initial perturbation at $t = 0$:

$$x(0) = x_0 + \xi(0),$$
$$\dot{x} = \dot{x}_0 + \dot{\xi} = \dot{\xi}$$
$$= f(x_0 + \xi) = f(x_0) + f'(x_0)\xi + O(\xi^2)$$

For an infinitesimal perturbation $\xi(0) = \xi_0$

$$\dot{\xi} = f'(x_0)\xi, \qquad f'(x_0) = \lambda, \qquad \dot{\xi} = \lambda\xi,$$
$$\Rightarrow \xi(t) = e^{\lambda t}\xi(0)$$

1. Fixed points, II

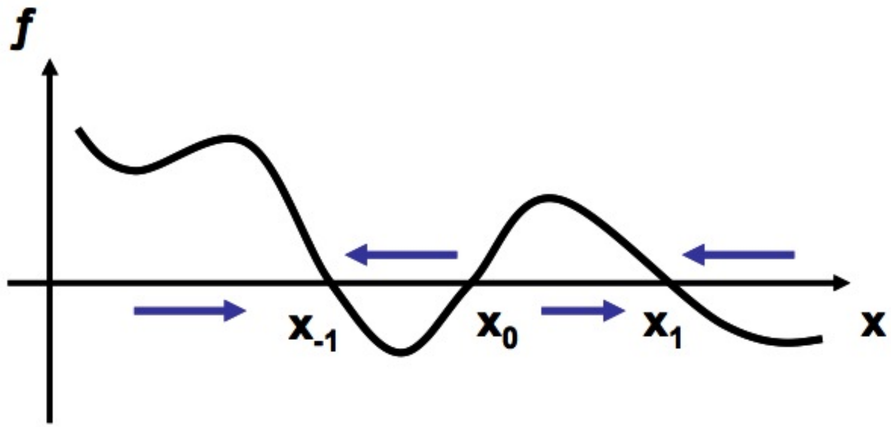
If $\lambda < 0 \Rightarrow$ the fixed point (FP) is (linearly) stable

If $\lambda > 0 \Rightarrow$ the FP is (linearly) unstable

If $\lambda = 0 \Rightarrow$ the linear stability of the FP is neutral

Some basic features on FPs:

1. $f \in C^1, f \not\equiv 0$ on all sub-intervals: FPs are isolated (generic property)
2. Basins of attraction are open intervals (possibly semi-infinite)



2. Saddle-node bifurcations

How does the geometry of the solutions change when $\mu \neq \mu_0$, i.e. how do the number of the stability of the stationary solution change?

Let us start with the scalar case.

A simple case: the saddle-node

$$\dot{x} = \mu - x^2 \equiv f(x;\mu)$$

FPs: $\mu - x^2 = 0$ $\quad x = \pm\sqrt{\mu}$

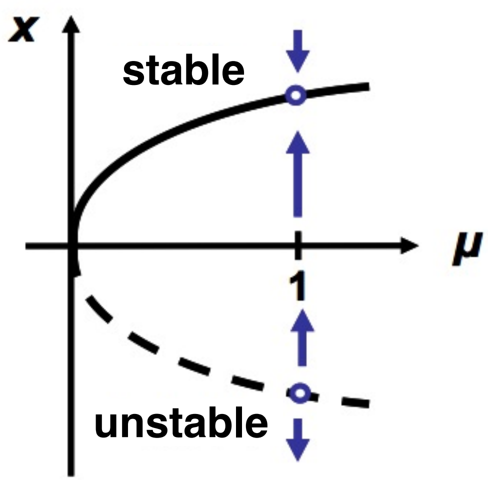
FP stability:

$$x_1 = \sqrt{\mu}, \quad x_{-1} = -\sqrt{\mu}$$

$$x(0) = x_{\pm 1} + \xi(0)$$

$$\dot{\xi} = \lambda_{\pm}\xi,$$

$$\lambda_{\pm} \equiv f'(x_{\pm 1}) = -2x_{\pm 1} = \mp 2\sqrt{\mu}$$



Upper branch (linearly) *stable*

Lower branch (linearly) *unstable*

Let us now examine the nonlinear stability

3. Bifurcations in *n*-D

We studied the scalar case ($n = 1$). More generally, we have:

$$\dot{\vec{x}} = \vec{f}(\vec{x};\mu)\,,\ \vec{f} \in C(\Re^n \times \Re)\,, \qquad \text{with } \vec{x} \in \Re^n \text{ and } \mu \in \Re.$$

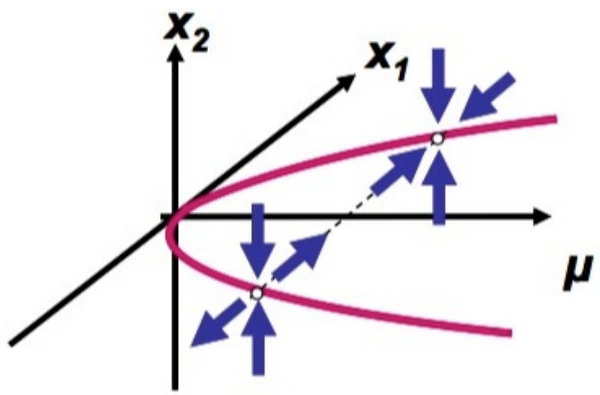
The behavior is "almost" linear in all the phase-parameter space $\Re^n \times \Re$, except in the neighborhood of a few isolated points (x_c, μ_c) : these are bifurcation points, where the Jacobian matrix $L = (\partial f_i / \partial x_j)$ is singular, i.e. $\det L = 0$

In the case n = 2, we can reduce to the normal form :

$$\dot{x_1} = \mu - x_1^2$$
$$\dot{x_2} = -\lambda x_2,\ \lambda > 0$$

In the general case, the reduction gives :

$$\dot{x_1} = \mu - x_1^2$$
$$\dot{x_i} = -\lambda_i x_i,\ \lambda_i > 0,\ i = 2, ..., n$$

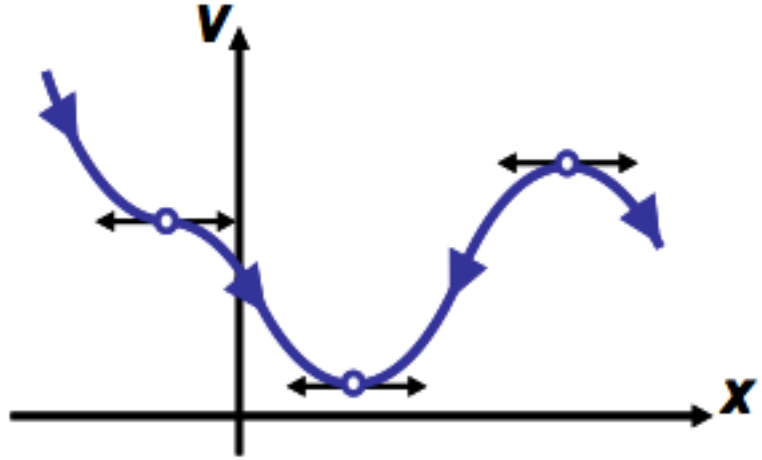


This shape explains the "saddle-node bifurcation" terminology

4. Non-linear stability and variational principle

To deepen our understanding of stability, we have to examine the effect of larger perturbations.

a) Variational principle in 0-D



$$V(x) = -\int_x f(\xi)\mathrm{d}\xi$$ — pseudo-potential

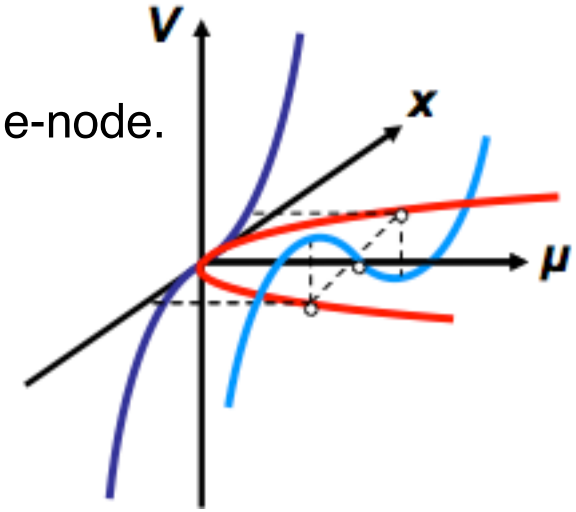
$$\dot{x} = f(x) = -V'(x)$$

$$\dot{x}^2 = -\frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = -\dot{V}$$

V will *decrease* along the ODE's trajectory as long as :

$$\dot{x} \neq 0 \Leftrightarrow V' \neq 0$$

$\dot{x} = 0$ if V reaches a minimum, a maximum, or a saddle-node.



Of course, only $V = min$ is stable — *nonlinearly*.

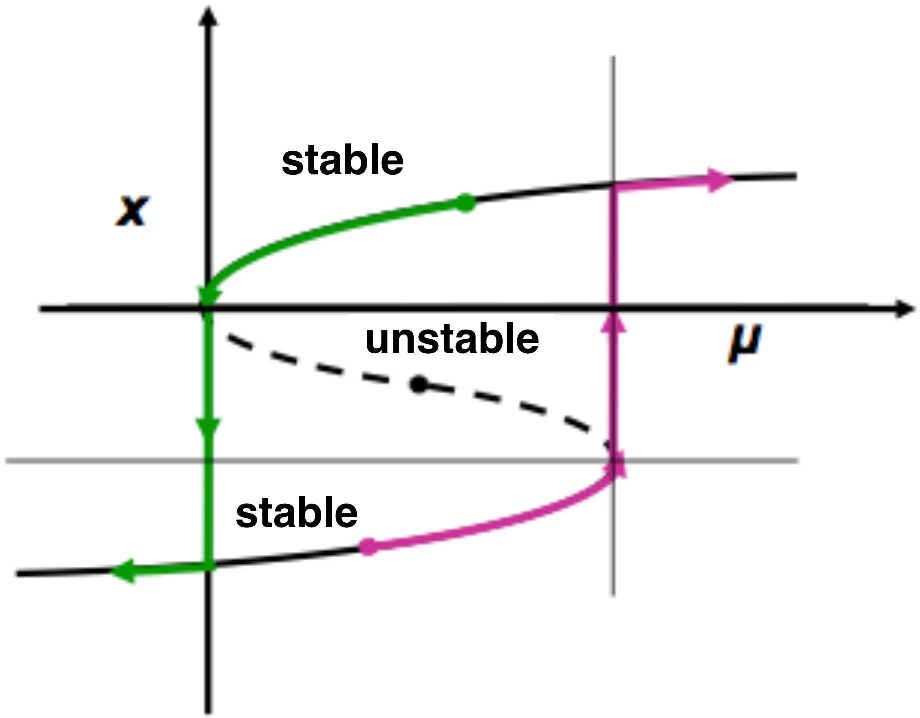
With this result, we turn back to the saddle-node bifurcation:

$$\dot{x} = \mu - x^2$$

$$V(x;\mu) = -\mu x + x^3/3 + c(\mu)$$

5. Bistability and hysteresis

The combination of two saddle-node bifurcations can create a hysteresis phenomenon (an *S*-shaped curve) :



$\dot{x} = \mu - x^2$: the top-left bifurcation

$\dot{x} = (\mu - 1) + (x + \frac{1}{2})^2$: the bottom-right bifurcation

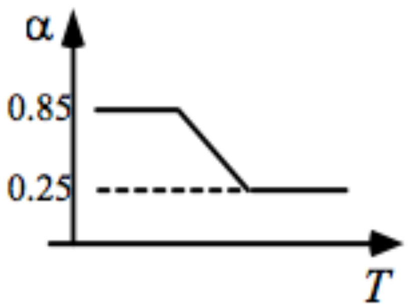
1-D version ('classic' EBM)

$$C(x)T_t = R_i - R_o + D$$

T — temperature
x — latitudinal coordinate
$\tilde{T}(x)$ — the observed climate

Boundary conditions: $T_x(0) = T_x(1) = 0$
$x = 0$ Pole (North)
$x = 1$ Equator

$$R_i = Q(x)\{1 - \alpha\}$$
$$= Q(x)\{1 - b(x) + c_1 T\}_c$$
$$R_o = \sigma T^4 \{1 - m \tanh(c_3 T^6)\}$$



$$D = \frac{1}{\sin \frac{\pi x}{2}} \partial_x \sin \frac{\pi x}{2} \{k(x) + k_s(x) g(\tilde{T})\} T_x$$

Questions: 1. Stationary solutions ('climates')?
2. Stability?
3. Perturbation & bifurcation? $Q \to \mu Q \quad (\mu = 1)$

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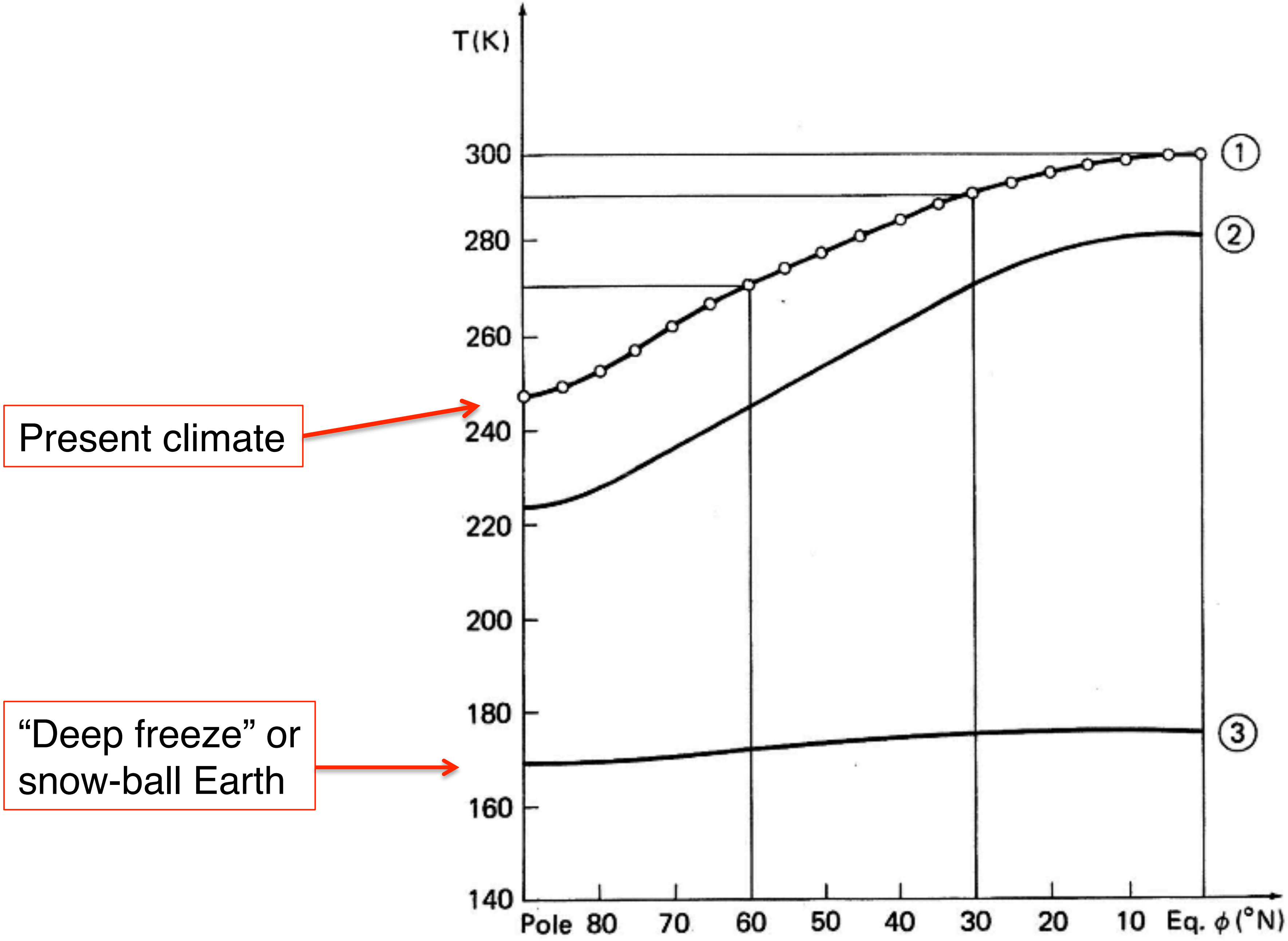
   - global (0-D) radiative equilibrium
   - latitude-dependent (1-D) radiative equilibrium
3. EBMs in 0-D (global)
   - equilibria & their linear stability
   - multiple stationary climates
4. Elementary bifurcation theory in 0-D
   - saddle-node bifurcations: multiple equilibria & their stability
   - nonlinear stability and variational principle
   - bistability and hysteresis
5. EBMs in 1-D (latitude-dependent)
   - applying all of the above
   - snowball Earth

The three climates of the 1-D model



1-D EBM: Bifurcation diagram

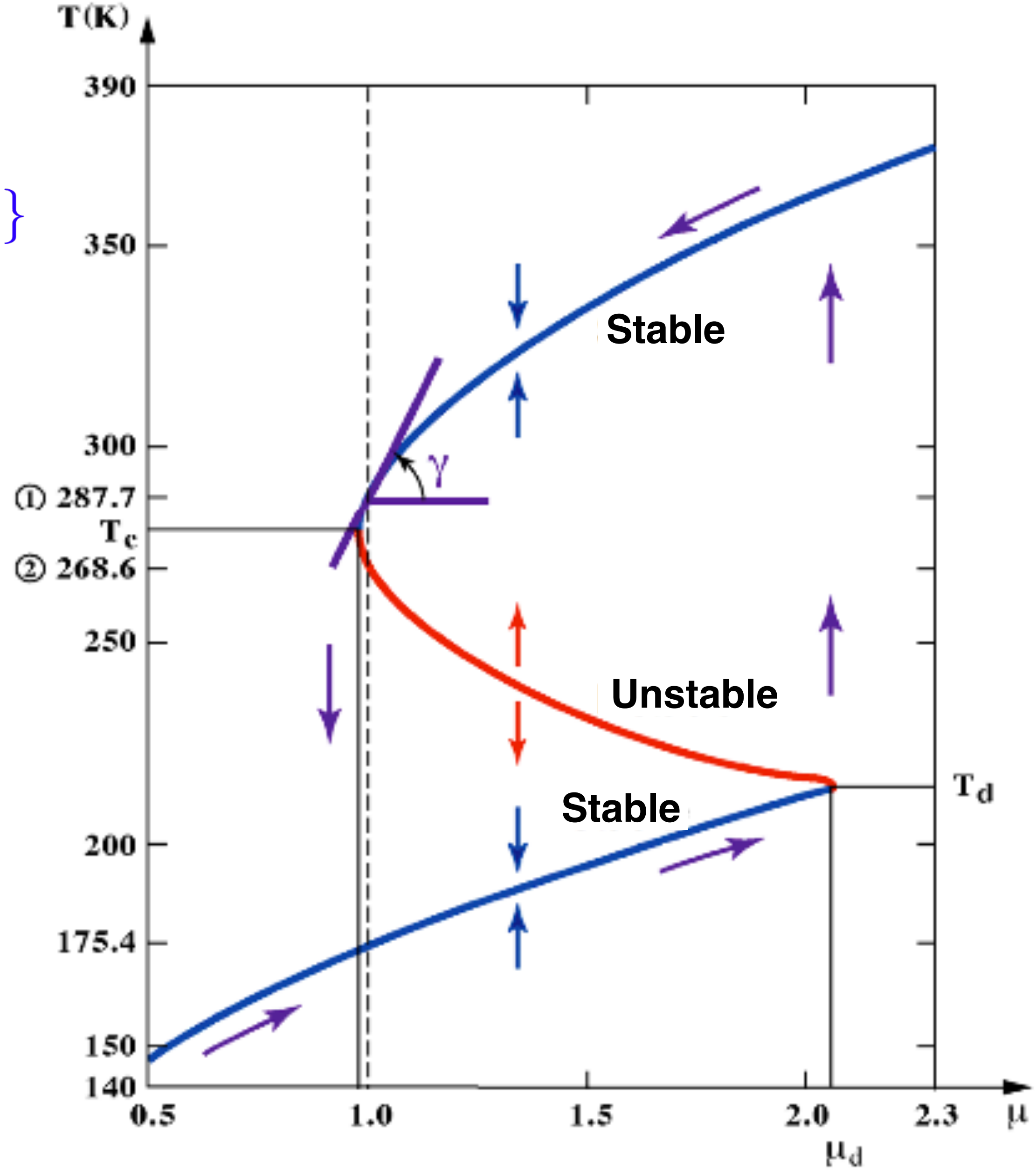
$$C(x)T_t = \{k(x,T)T_x\}$$
$$+\mu Q_0\{1-\alpha(x,T)\}$$
$$-g(T)\sigma T^4$$

$T_x = 0$ at $x = 0, 1$

Climate sensitivity:

$$\gamma = \frac{dT}{d\mu} \cong 0.01$$

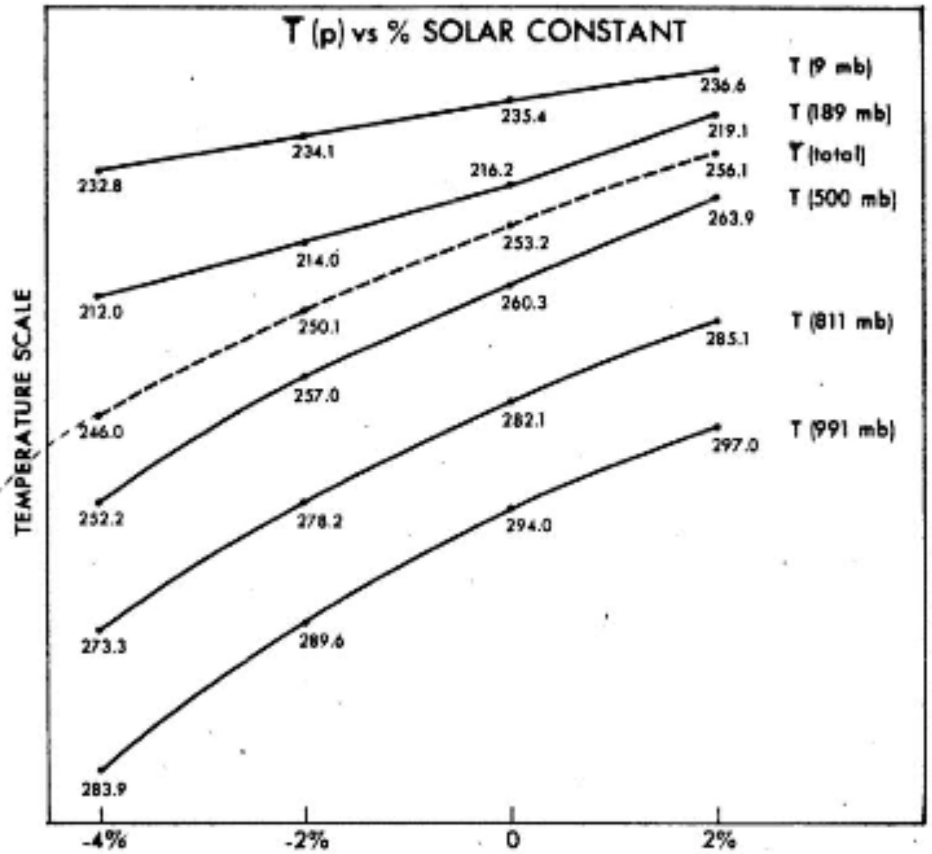
(1K per % of Q)



Climate sensitivity to insolationin in a General Circulation Model (GCM)

"As stated in the Introduction, it is not, however, reasonable to conclude that the present results are more reliable than the results from the one-dimensional studies mentioned above simply because our model treats the effect of transport explicitly rather than by parameterization."*

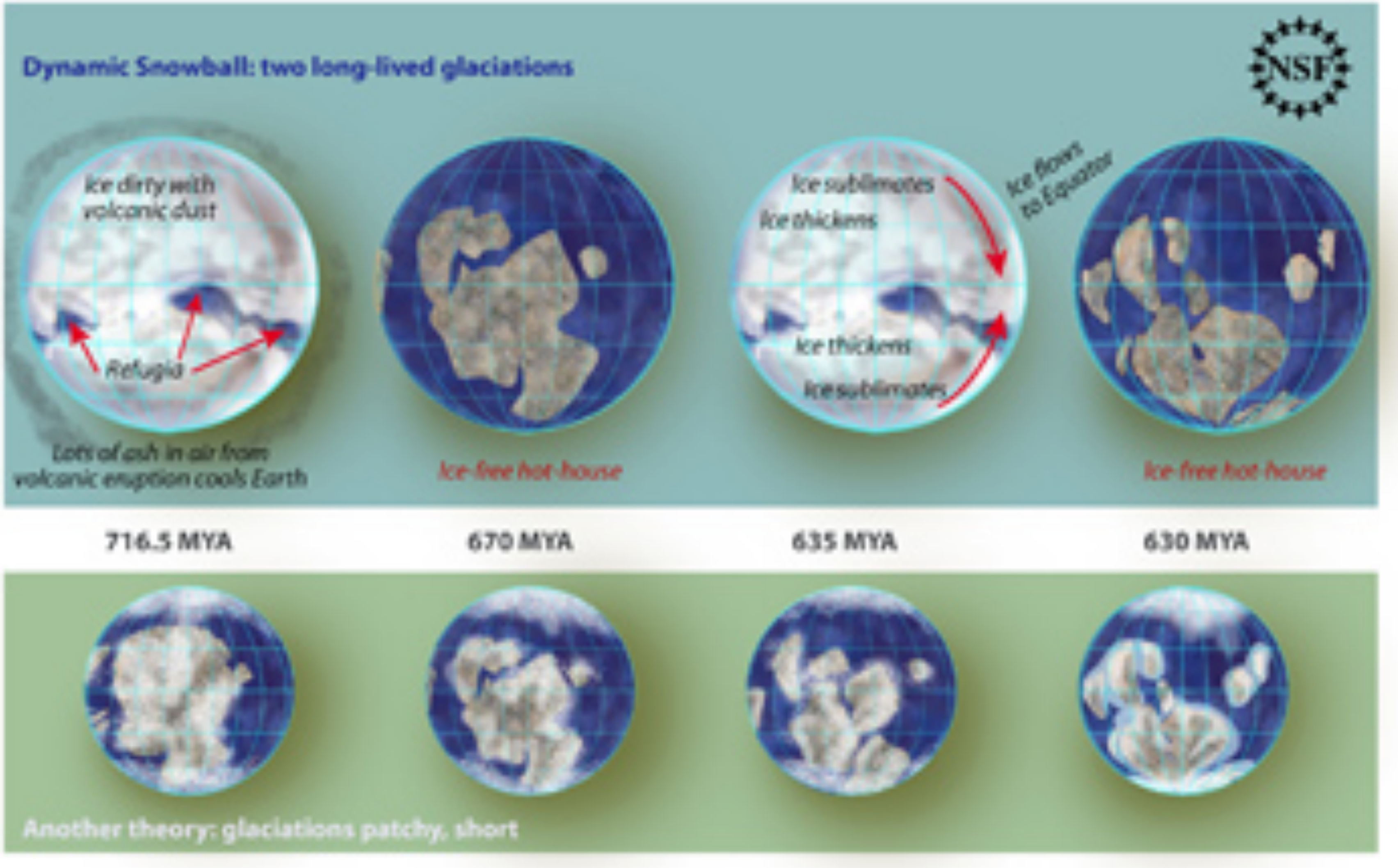
"Nevertheless, it seems to be significant that both the one-dimensional and three- dimensional models yields qualitatively similar results in many respects."*



Area-mean temperatures for various model levels, as well as a mass-weighted mean temperature for the total model atmosphere. Based on 4 GCM runs: control, –4%, –2% and +4%. Units are in degrees K.

* From Wetherald and Manabe, 1975, *J. Atmos. Sci.*, **32,** 2044–2059.

Snowball Earth — Erstwhile a "theory"; now a "fact"?



https://www.nsf.gov/news/news_images.jsp?cntn_id=116410&org=NSF

Distance to "tipping points"?

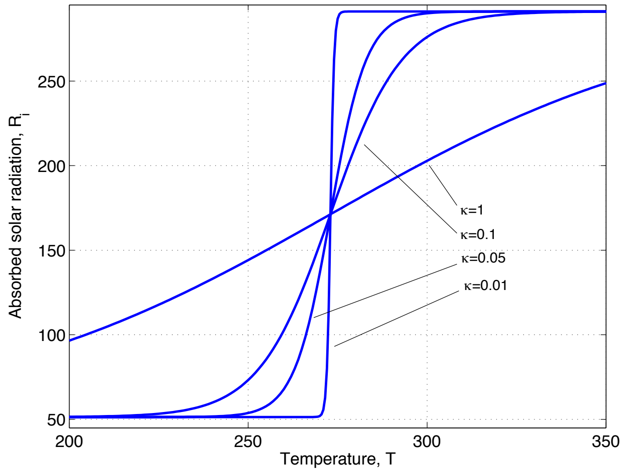
Slightly modified 0-D EBM (Zaliapin & Ghil, *NPG, 2010)*

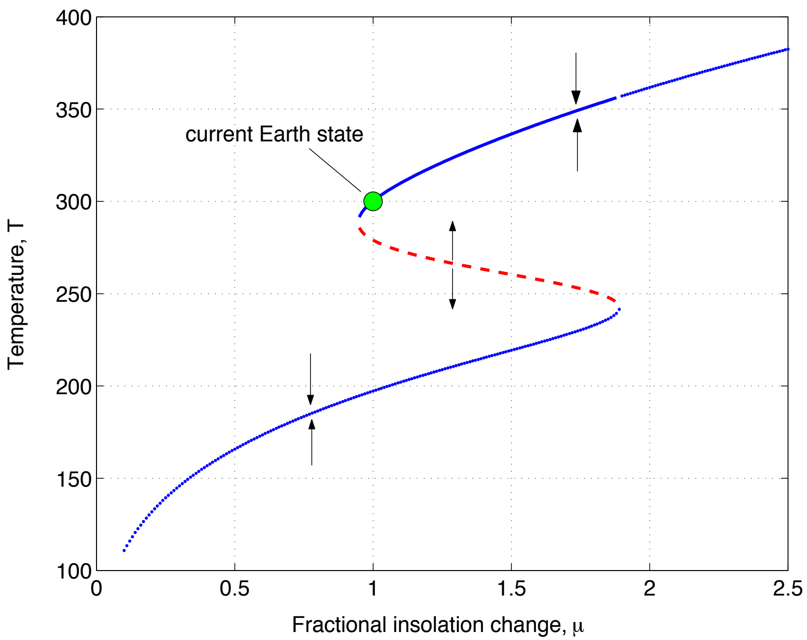
$$c\dot{T} = \mu Q_0(1 - \alpha(T))\sigma T^4[1 - m\tanh((T/T_0)^6)]$$

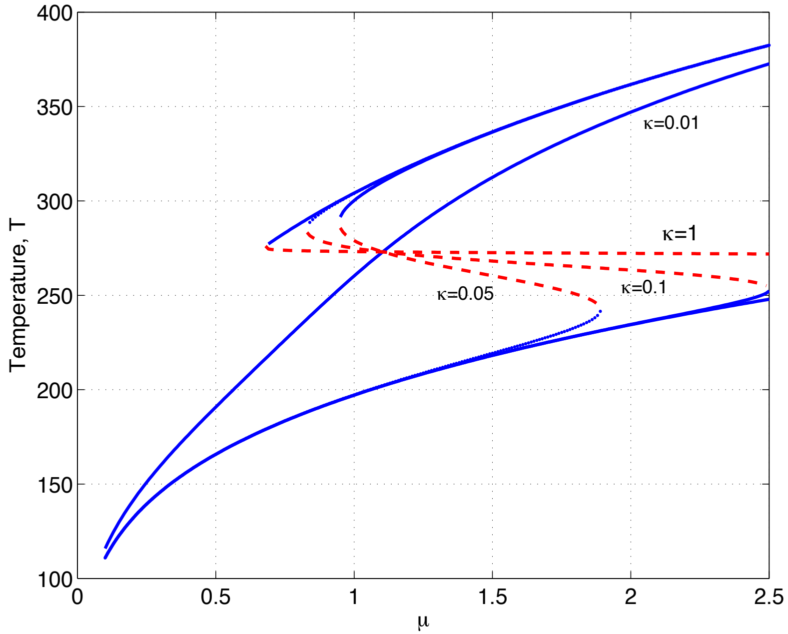
$$\alpha(T;\kappa) = c_1 + c_2\frac{1 - \tanh[\kappa(T - T_c)]}{2}$$

T_c is the ice-margin temperature,

while κ is an extra "Budygo-vs.-Sellers" parameter







Double-well potential in 2-D

1-D EBM of Budyko-Sellers-North, cf. Held & Suarez (*Tellus,* 1974); North *et al.* (*JAS*, 1979).

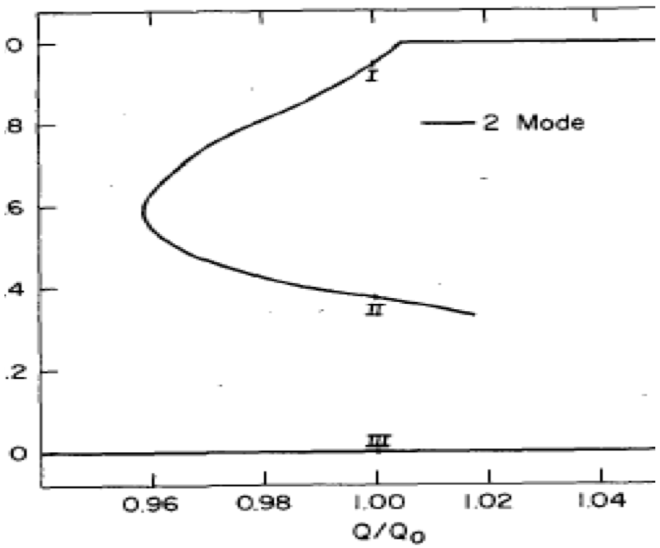
Taking $x = \sin(latitude)$ and $k(x,T) = k_0$,
We get the semi-linear parabolic PDE

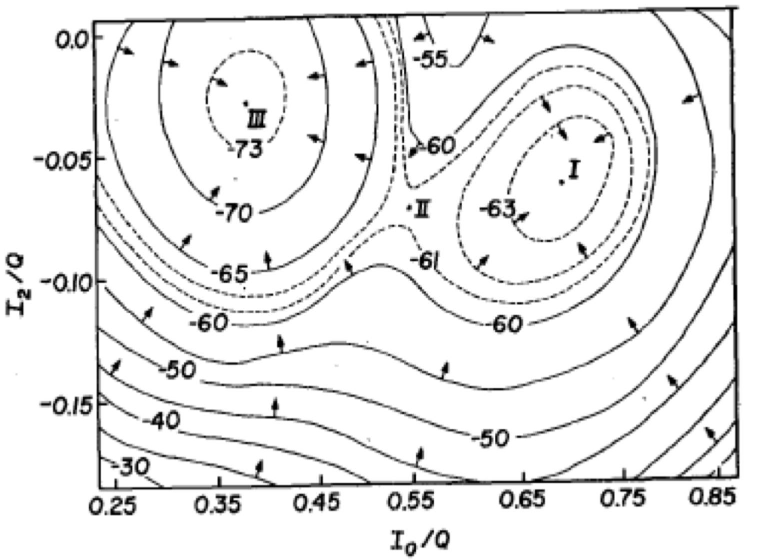
$$CT_t = [k_0(1-x^2)T_x]_x + Q(x)[1-\alpha(T)] - I(T)$$

which yields the variational principle:

$$F\{T(x)\} = \int \{\frac{1}{2}k_0(1-x^2)T_x^2 - Q(x)A(T) + R(T)\}\,dx,$$ where

$$A(T) = \int^T [1-\alpha(T)]\,dT,$$ and $$R(T) = \int^T I(T)\,dT.$$





Concluding remarks, I

- Tipping points and bifurcations: multiple equilibria and rapid transitions between them.
- Prediction of the transitions? To follow.
- Transitions between more general types of behavior — limit cycles, strange attractors — likewise to follow.

Some general references

Andronov, A.A., and L.S. Pontryagin, 1937: Systèmes grossiers. *Dokl. Akad. Nauk. SSSR*,**14**(5), 247–250.

Arnold, L., 1998: *Random Dynamical Systems*, Springer Monographs in Math., Springer, 625 pp.

Arnol'd, V. I., 1983: *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York/Heidelberg/Berlin, 334 pp.

Charney, J.G., *et al*., 1979: *Carbon Dioxide and Climate: A Scientific Assesment*. National Academic Press, Washington, D.C.

Dijkstra, H.A., 2005: *Nonlinear Physical Oceanography : A Dynamical Systems Approach to the Large-Scale Ocean Circulation and El Niño*, 2nd edn., Springer, 532 pp.

Dijkstra, H. A., and M. Ghil, 2005: Low-frequency variability of the large-scale ocean circulation: A dynamical systems approach, *Rev. Geophys.,* **43**, RG3002, doi:10.1029/2002RG000122.

Ghil, M., R. Benzi, and G. Parisi (Eds.), 1985: *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, North-Holland, 449 pp.

Ghil, M., and S. Childress, 1987: *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory and Climate Dynamics*, Ch. 5, Springer-Verlag, New York, 485 pp.

Ghil, M., M.D. Chekroun, and E. Simonnet, 2008: Climate dynamics and fluid mechanics: Natural variability and related uncertainties, *Physica D*, **237**, 2111–2126.

Lorenz, E.N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130–141.

Ruelle, D., and F. Takens, 1971: On the nature of turbulence. *Commun. Math. Phys.*, **20**, 167–192.

Solomon, S., *et al*. (Eds.). *Climate Change 2007: The Physical Science Basis*. *Contribution of Working Group I to the Fourth Assessment Report of the IPCC*, Cambridge Univ. Press, 2007.

Reserve slides

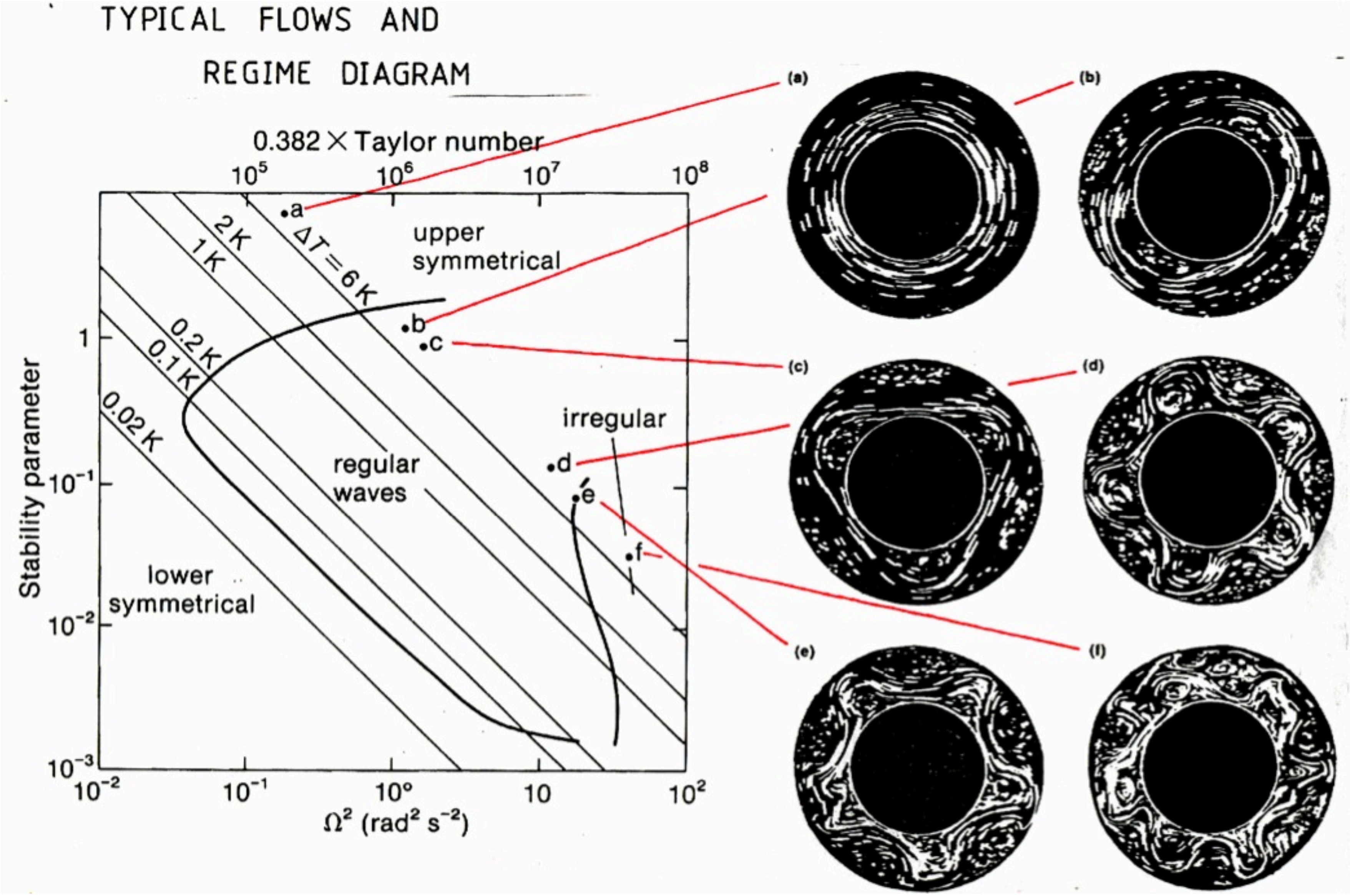
Motivation

- There's a lot of talk about "tipping points."
- It sounds threatening, like falling off a cliff: that's why we care!
- But what are they, and what do **we know** about them?
- Here's a **disambiguation page** (cf. Wikipedia), first.
- *Sociology*: "the moment of critical mass, the threshold, the boiling point" (Gladwell, 2000); a previously rare phenomenon becomes rapidly and dramatically more common.
- *Physics*: the point at which a system changes from a stable equilibrium into a new, qualitatively dissimilar equilibrium (throwing a switch, tilting a plank, boiling water, etc.).
- *Climatology*: "A climate tipping point is a somewhat ill-defined concept […]"— so we'll try to actually define it better.

- *Catastrophe theory*: branch of bifurcation theory in the study of dynamical systems; here, a tipping point is "a parameter value at which the set of equilibria abruptly change." ➔ **Let's see!**

M. Gladwell (2000) *The Tipping Point: How Little Things Can Make a Big Difference.*
T. M. Lenton *et al.* (2008) Tipping elements in the Earth's climate system, *PNAS,* v. **105.**

Rotating Convection: An Illustration



M. Ghil, P.L. Read & L.A. Smith (*Astron. Geophys., 2010)*

Energy Balance Models (EBMs)

Budyko, Sellers and Held-Suarez-North

10.2. Energy-Balance Models (EBMs): Multiple Equilibria 311

Table 10.1. Comparison of Budyko's and Sellers' models.

Heat Flux	Budyko	Sellers
$R_i = Q(1 - \alpha(T))$ Absorbed solar radiation, as a function of ice-albedo feedback	Step-function albedo $\alpha = \begin{cases} \alpha_M, & T < T_s, \\ \alpha_m, & T \geq T_s, \end{cases}$ $\alpha_M > \alpha_m$, $T_\ell \leq T_s \leq T_u$	Ramp-function albedo $\alpha = \begin{cases} \alpha_M, & T < T_\ell \\ \alpha_M - \frac{T-T_\ell}{T_u-T_\ell}(\alpha_M - \alpha_m) & T_\ell \leq T < T_u, \\ \alpha_m, & T_u \leq T \end{cases}$
R_0 Outgoing IR radiation	Linear, empirical $A + BT$	Stefan-Boltzmann law with greenhouse effect $\sigma T^4\{1 - m \tanh(T^6/T_0^6)\}$
$\nabla \cdot F$ Horizontal flux divergence	Newtonian cooling $\kappa\{T(\phi) - \overline{T}\}$	Eddy-diffusive $\nabla \cdot \{k(\phi)\nabla T(\phi)\}$

2[nd] column:
Budyko (1968, 1969)

3[rd] column:
Sellers (1969)

In red:
the "mixed" version of
Held & Suarez (1974)
and North (1975a, b)